Project # 1

1. Prove that the metric space $(C([0,1]), d)$ is complete, where the metric d is defined as

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

2. Prove that the metric space $(C([0,1]), d)$ is not complete, where the metric d is defined as

$$d(f,g) = \int_0^1 |f(x) - g(x)|\,\mathrm{d}x.$$

Hint: Consider the follownig sequence $\{f_n \in C([0,1])\colon n \in \mathbb{N}_{\geq 1}\}$

$$f_n(x) = \begin{cases} 0 & x \in [0, \frac{1}{3}] \\ \text{linear} & x \in [\frac{1}{3}, \frac{1}{3} + \frac{2}{3}\cdot\frac{1}{2^n}] \\ 1 & x \in [\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{2^n}, 1] \end{cases}$$